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## The natural operators of general affine connections into general affine connections

ABSTRACT. We reduce the problem of describing all  $\mathcal{M}f_m$ -natural operators transforming general affine connections on  $m$ -manifolds into general affine ones to the known description of all  $GL(\mathbf{R}^m)$ -invariant maps  $\mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$  for  $k = 1, 3$ .

**Introduction.** All manifolds considered in this paper are assumed to be finite dimensional, without boundaries, second countable, Hausdorff and smooth (of class  $C^\infty$ ). Maps between manifolds are assumed to be smooth (of class  $C^\infty$ ). The category of  $m$ -dimensional manifolds and their embeddings is denoted by  $\mathcal{M}f_m$ .

A classical linear connection on a manifold  $M$  is a right invariant connection  $\Gamma$  on the principal fiber bundle  $LM$  of linear frames of  $M$ . It can be considered equivalently as the corresponding  $\mathbf{R}$ -bilinear map  $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$  such that  $\nabla_{fX}Y = f\nabla_XY$  and  $\nabla_XfY = X(f)Y + f\nabla_XY$  for any map  $f : M \rightarrow \mathbf{R}$  and any vector fields  $X, Y \in \mathcal{X}(M)$  on  $M$ , see [2].

A general affine connection on  $M$  is a right invariant connection  $\Gamma$  on the principal fiber bundle  $AM$  of affine frames of  $M$ . It can be equivalently considered as the corresponding pair  $(\nabla, K)$  consisting of a classical linear connection  $\nabla$  on  $M$  and a tensor field  $K$  of type  $(1, 1)$  on  $M$ , see [2].

The general concept of natural operators can be found in [3].

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2010 *Mathematics Subject Classification.* 58A32.

*Key words and phrases.* General affine connection, natural operator.

In the present note, we study the problem of finding all  $\mathcal{M}f_m$ -natural operators  $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$  transforming general affine connections  $(\nabla, K)$  on  $m$ -manifolds  $M$  into general affine connections  $B(\nabla, K)$  on  $M$ .

Given an  $\mathcal{M}f_m$ -natural operator  $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$ , we define an  $\mathcal{M}f_m$ -natural operator  $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$  by

$$B(\nabla, K) = (\nabla, K) + \Delta(\nabla, K)$$

for all general affine connections  $(\nabla, K)$  on  $m$ -manifolds  $M$ , and vice versa. So, to find all  $\mathcal{M}f_m$ -natural operators  $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$  it is sufficient to find all  $\mathcal{M}f_m$ -natural operators  $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$  transforming general affine connections  $(\nabla, K)$  on  $m$ -manifolds  $M$  into pairs  $\Delta(\nabla, K) = (\Delta^1(\nabla, K), \Delta^2(\nabla, K))$  of tensor fields  $\Delta^1(\nabla, K)$  of type  $(1, 2)$  and  $\Delta^2(\nabla, K)$  of type  $(1, 1)$  on  $M$ .

In the present note, we prove that the above problem of finding all  $\mathcal{M}f_m$ -natural operators  $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$  (or  $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ ) can be reduced to the one of describing all  $GL(\mathbf{R}^m)$ -invariant maps  $\mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$  for  $k = 1, 3$ .

This “reduction” is satisfactory, because the  $GL(\mathbf{R}^m)$ -invariant maps  $\mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$  for  $k = 1, 2, 3$  are described in [1].

**1. The crucial lemma.** We prove the following lemma.

**Lemma 1.** *There is the bijection between the set  $C$  of all  $\mathcal{M}f_m$ -natural operators  $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$  and the set  $D$  of all  $GL(\mathbf{R}^m)$ -invariant maps  $(\wedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \rightarrow (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$ .*

**Proof.** We define a map  $\Phi : C \rightarrow D$  as follows.

Any  $\Delta \in C$  is determined by the values

$$\begin{aligned} \Delta(\nabla, K)(x) &= (\Delta^1(\nabla, K)(x), \Delta^2(\nabla, K)(x)) \\ &\in (\otimes^2 T_x^* M \otimes T_x M) \oplus (T_x^* M \otimes T_x M) \end{aligned}$$

for all  $m$ -manifolds  $M$ , all linear connections  $\nabla$  on  $M$ , all tensor fields  $K$  of type  $(1, 1)$  on  $M$  and all  $x \in M$ . Because of the  $\mathcal{M}f_m$ -invariance of  $\Delta$ , we may assume that  $M = \mathbf{R}^m$ ,  $x = 0$ . We can even assume that  $id_{\mathbf{R}^m}$  is  $\nabla$ -normal with center 0 (then  $\nabla(0) \in \wedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m$  because the Christoffel symbols  $\nabla_{jk}^i$  of  $\nabla$  satisfy  $\nabla_{jk}^i(0) + \nabla_{kj}^i(0) = 0$ ). Then using the invariance of  $\Delta$  with respect to the homotheties  $a_t = t id_{\mathbf{R}^m}$  for  $t > 0$ , we obtain the homogeneity condition

$$\Delta((a_t)_* \nabla, (a_t)_* K)(0) = (t \Delta^1(\nabla, K)(0), \Delta^2(\nabla, K)(0)) .$$

Because of the homogeneous function theorem [3], this type of the homogeneity implies that  $\Delta(\nabla, K)(0)$  depends on  $\nabla(0)$  and  $j_0^1 K$  (only). Let  $(\Lambda, \tau_0, \tau_1) \in (\wedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \simeq (\wedge^2 T_0^* \mathbf{R}^m \otimes$

$T_0\mathbf{R}^m) \oplus J_0^1(T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$ , where  $\hat{=}$  is the usual  $GL(\mathbf{R}^m)$ -invariant identification. We put

$$\Phi(\Delta)(\Lambda, \tau_0, \tau_1) := \Delta(\nabla, K)(0) \in (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$$

(modulo the usual  $GL(\mathbf{R}^m)$ -invariant identification), where  $\nabla$  is the linear connection on  $\mathbf{R}^m$  such that the Christoffel symbols of  $\nabla$  with respect to the chart  $id_{\mathbf{R}^m}$  are constant maps and  $\nabla(0) = \nabla^o(0) + \Lambda$  and  $\nabla^o$  is the usual flat torsion free connection on  $\mathbf{R}^m$  and  $K$  is the tensor field of type  $(1, 1)$  on  $\mathbf{R}^m$  such that the coefficients of  $K$  in the chart  $id_{\mathbf{R}^m}$  are polynomials of degree not more than 1 and  $j_0^1 K = (\tau_0, \tau_1)$ .

Since  $\Delta$  is determined by  $\Phi(\Delta)$ ,  $\Phi$  is injective.

It remains to show that  $\Phi$  is surjective. Let  $c : (\bigwedge^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \rightarrow (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$  be a  $GL(\mathbf{R}^m)$ -invariant map (an element from  $D$ ). Using the usual  $GL(\mathbf{R}^m)$ -invariant identification  $\mathbf{R}^m = T_0\mathbf{R}^m$ , we have the  $GL(\mathbf{R}^m)$ -invariant map

$$\begin{aligned} c : \left( \bigwedge^2 T_0^*\mathbf{R}^m \otimes T_0\mathbf{R}^m \right) \oplus \left( J_0^1(T^*\mathbf{R}^m \otimes T\mathbf{R}^m) \right) &\rightarrow \\ &\rightarrow (\otimes^2 T_0^*\mathbf{R}^m \otimes T_0\mathbf{R}^m) \oplus (T_0^*\mathbf{R}^m \otimes T_0\mathbf{R}^m). \end{aligned}$$

Let  $(\nabla, K)$  be a general connection on an  $m$ -manifold  $M$ . Using  $c$ , we define a pair  $\Delta_c(\nabla, K)$  consisting of tensor fields  $\Delta_c^1(\nabla, K)$  of type  $(1, 2)$  and  $\Delta_c^2(\nabla, K)$  of type  $(1, 1)$  on  $M$  as follows. Let  $x \in M$ . Consider a normal coordinate system  $\varphi$  of  $\nabla$  with center  $x$ . Then  $(\varphi_*\nabla)_0 \in \bigwedge^2 T_0^*\mathbf{R}^m \otimes T_0\mathbf{R}^m$  modulo the obvious  $GL(\mathbf{R}^m)$ -invariant identification and  $j_0^1(\varphi_*K) \in J_0^1(T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$ . We put

$$(\varphi_*\Delta_c(\nabla, K))_0 := c((\varphi_*\nabla)_0, j_0^1(\varphi_*K)).$$

If  $\psi$  is another normal coordinate system of  $\nabla$  with center  $x$ , then  $\psi = \eta \circ \varphi$  for a  $GL(\mathbf{R}^m)$ -map  $\eta$ . Then  $(\psi_*\Delta_c(\nabla, K))_0 = (\varphi_*\Delta_c(\nabla, K))_0$  because of the  $GL(\mathbf{R}^m)$ -invariance of  $c$ . That is why, the definition of  $\Delta_c(\nabla, K)$  is correct. Thus we have the  $\mathcal{M}f_m$ -natural operator  $\Delta_c : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ . Clearly,  $\Phi(\Delta_c) = c$ .  $\square$

**2. The main result.** The main result of the note is the following “reduction” theorem.

**Theorem 1.** *The problem of finding all  $\mathcal{M}f_m$ -natural operators  $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$  can be reduced to the one of describing all  $GL(\mathbf{R}^m)$ -invariant maps  $\mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$  for  $k = 1, 3$ .*

**Proof.** Any  $GL(\mathbf{R}^m)$ -invariant map  $c : (\bigwedge^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \rightarrow (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$  is the system of  $GL(\mathbf{R}^m)$ -invariant maps

$$c_1 : \left( \bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m \right) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \rightarrow \otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m$$

and

$$c_2 : \left( \bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m \right) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \rightarrow \mathbf{R}^{m*} \otimes \mathbf{R}^m.$$

Using the invariance of  $c_i$  with respect to the homotheties  $a_t = tid_{\mathbf{R}^m}$  for  $t > 0$ , we obtain the respective homogeneity conditions. Then (by the homogeneous function theorems)  $c_1(\Lambda, \tau_0, \tau_1)$  is linear in  $\Lambda$  and  $\tau_1$  and not necessarily linear in  $\tau_0$ . Then  $c_1$  can be treated as the sum of  $GL(\mathbf{R}^m)$ -linear maps

$$c'_1 : \mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \left( \bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m \right)^* \otimes (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \subset \otimes^3 \mathbf{R}^{m*} \otimes \otimes^3 \mathbf{R}^m$$

and

$$c''_1 : \mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m)^* \otimes (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \simeq \otimes^3 \mathbf{R}^{m*} \otimes \otimes^3 \mathbf{R}^m.$$

By the same arguments,  $c_2(\Lambda, \tau_0, \tau_1)$  is independent of  $\Lambda$  and  $\tau_1$ . Then  $c_2 : \mathbf{R}^{m*} \otimes \mathbf{R}^m \rightarrow \mathbf{R}^{m*} \otimes \mathbf{R}^m$  is a  $GL(\mathbf{R}^m)$ -invariant map.

Now, Theorem 1 is an immediate consequence of Lemma 1.  $\square$

#### REFERENCES

- [1] Dębecki, J., *The natural operators transforming affinors to tensor fields of type (3, 3)*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **39** (2000), 37–49.
- [2] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry. Vol. I*, J. Wiley-Interscience, New York–London, 1963.
- [3] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

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Received December 31, 2016